Some remarks on the question of pseudocylindrical projections with minimum distortions for world maps

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- Traditional expectations of map projections:
  - symmetry
  - scale invariance
  - similarity
- larger territory to be represented - stronger map distortions; requirement for *measurability* is forced back requirement for *illustrativity* comes to the front
- In case of representation of the whole Earth the illustrativity refers

  - to the similarity of the *Earth objects* (continents and oceans)

  - to the similarity of *the Earth as a whole*
- The projection of a small scale map, mainly of a world map is proper in that case, if its distortions don’t hinder the illustrativity of the map

North Amerika in equal-area projection: *stretch*

North Amerika in conformal projection: *area ratio turnover*

- *Area* and *angular* distortions „effect against one another”
- Their extreme rising can be prevented by application of *aphylactic* (neither equal area, nor conformal) projections
- This kind of projections keep the balance between the two distortions, so the illustrative feature of the map remains
- (Remark: the *scale distortion* has an other nature, and can be originated in the area and angular distortions)

- Kavrayskiy type measure of distortions in question (using the ***a*** maximal and ***b*** minimal linear scales):
- $\mathbf{\ln^2(a \cdot b)}$ - index number which characterizes the *local area distortion*; in case of equivalency equals zero
- $\mathbf{\ln^2(a/b)}$ - index number which characterizes the *local angular distortion*; in case of conformity equals zero
- The mean value of this parameters gives an index number for the *local* overall distortions (*in a point* of the map)

$$\varepsilon_K{}^2 = \frac{\ln^2(a \cdot b) + \ln^2\left(\frac{a}{b}\right)}{2} = \ln^2 a + \ln^2 b$$

- In case of *simultaneously distorted area and angles* this value can be smaller than in case of equivalency or conformity

- The average of this index number (the mean overall distortion, so called *Airy-Kavrayskiy criterion*) gives the distortions for the territory T to be represented on the whole (*globally*) :

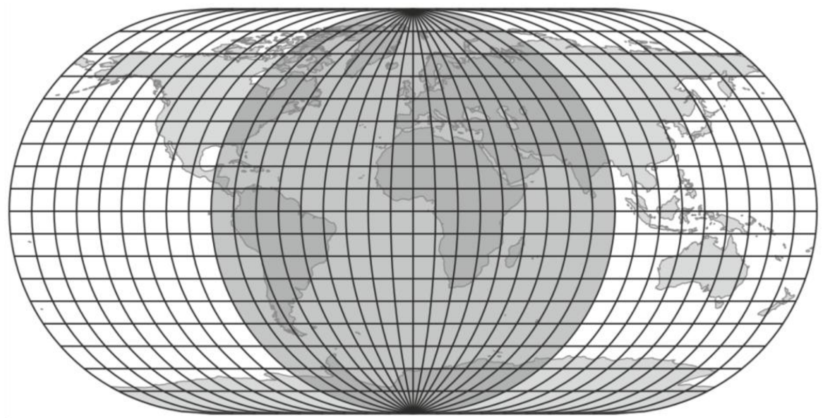
$$E_K{}^2 = \frac{1}{\mu(T)} \cdot \int_T \varepsilon_K{}^2 \, dT$$

- In case of representing the whole Earth, the 5° environments of the poles have to be excluded because of the extremely huge distortions:

$$E_K{}^2 = \frac{1}{2 \cdot \sin 85°} \cdot \int_{-85°}^{85°} \int_{-180°}^{180°} \varepsilon_K{}^2 \cdot \cos\varphi \, d\lambda \, d\varphi$$

- The *smaller* is this criterion value the *more favourable* is the projection from the aspect of distortions

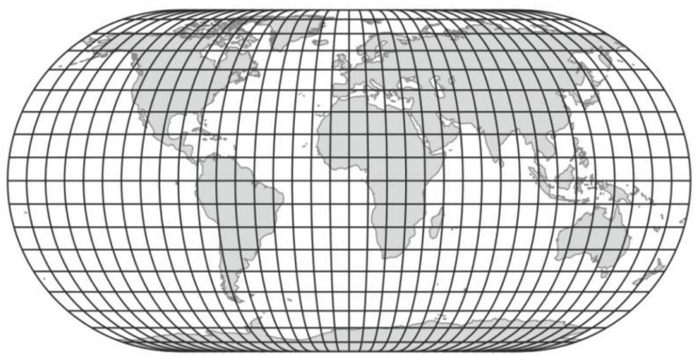
- *Pseudocylindrical* projections: the lines of latitude appear as *parallel straight lines* on the map. They are often used for representation of the whole Earth



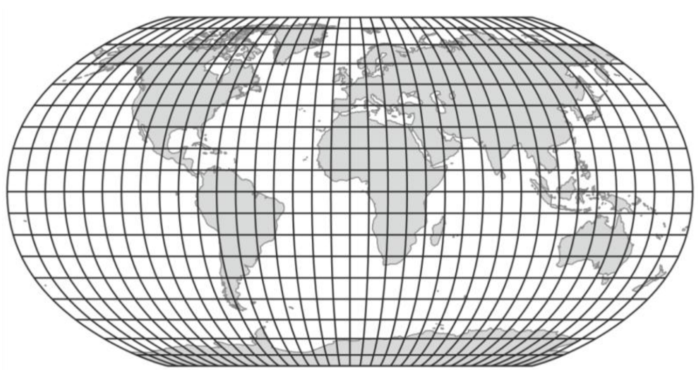
- If the pole is represented as a straight line („flat-polar" projection): <u>more favourable distortions</u>
- The outline of the Earth on the map in such a projection can not reflect the oval shape of the Earth

„flat-polar” projections

Eckert IV.

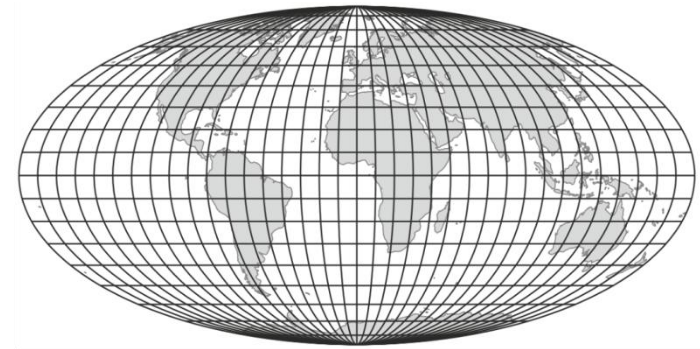


Robinson

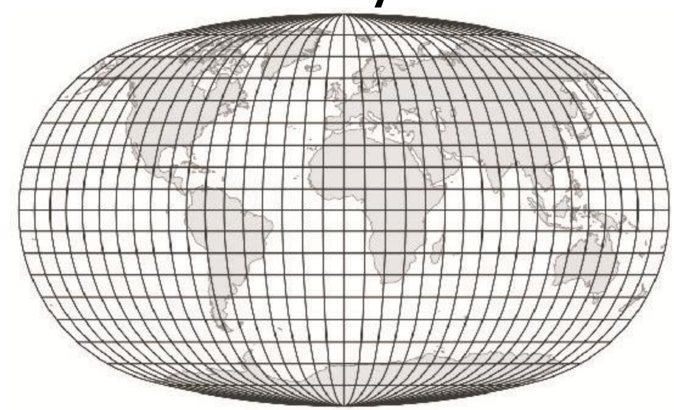


„pointed-polar” projections

Mollweide



Baranyi IV.



If the pole is represented as a point („pointed-polar” projection): the oval shape of the mapped Earth can be kept

- The projections are described by functions („projection equations”*)* which give the map coordinates (x,y) of any point from its geographic coordinates (φ,λ).
- The main task was *the determination of the projection equations* $y(\varphi)$ and $x(\varphi,\lambda)$ of the demanded pseudocylindrical projections by the *minimization* of the Airy-Kavrayskiy criterion.
- The optimization of the functions $y(\varphi)$ and $x(\varphi,\lambda)$ was calculated *separately*, with the help of calculus of variations.

- The optimal projection equation $y(\varphi)$ can be derived from the solution of a differential equation.
- The solution is - independently of the function $x(\varphi,\lambda)$ and the territory to be represented -

$$dy/d\varphi = 1, \quad \text{so} \quad y=\varphi$$

  which results in a *true scale midmeridian*.
- It seems trivial, however some of often applied pseudocylindrical projections, e.g. Robinson projection have not this feature.

The optimized projection equation $x(\varphi,\lambda)$ was originated from the product

$$\left[1-\left(\frac{2\cdot|\varphi|}{\pi}\right)^{c_2}\right]^{\frac{1}{c_3}}\cdot\lambda$$

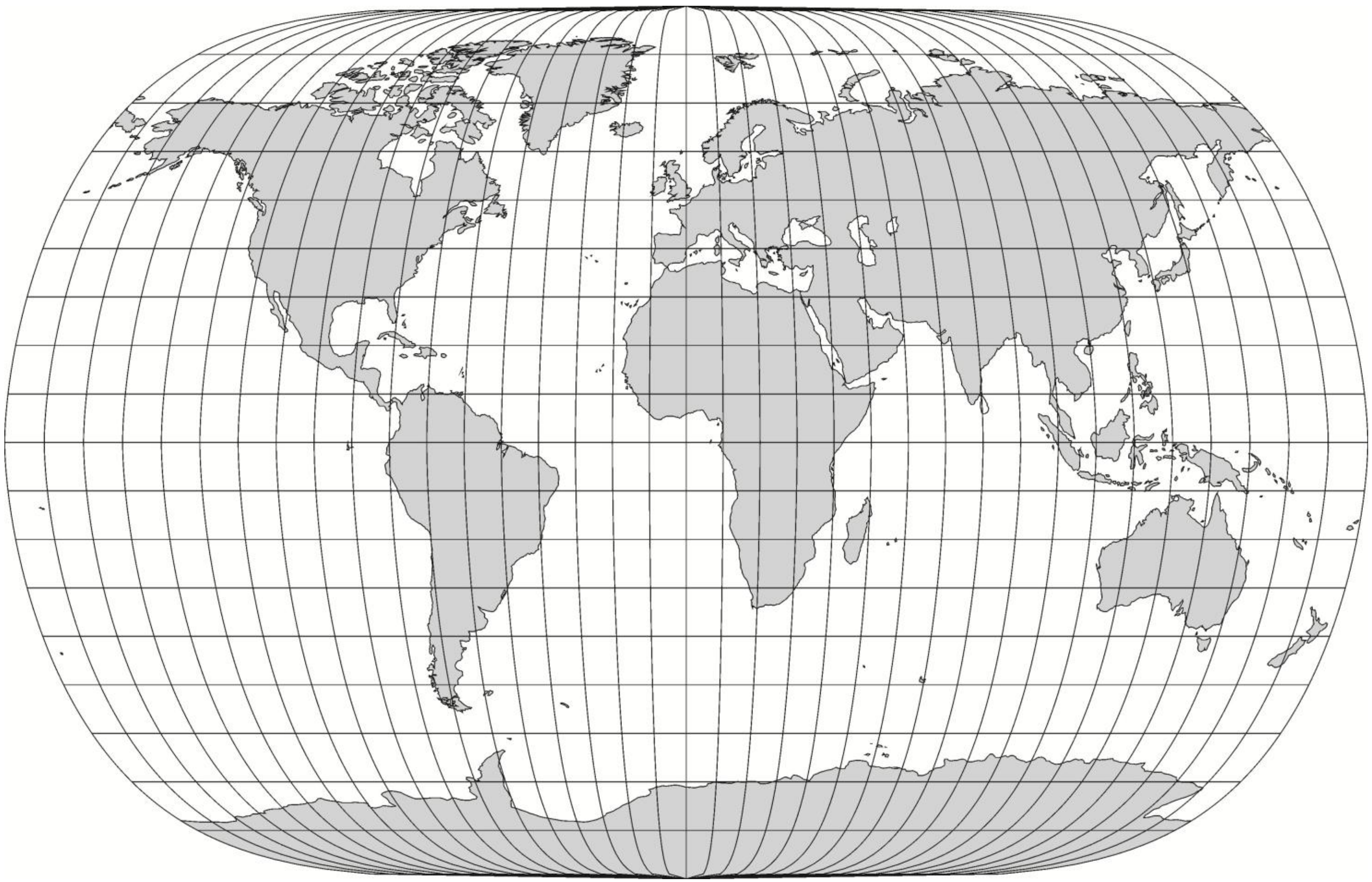
(c_2 and c_3 regulate the running down of the meridians.)

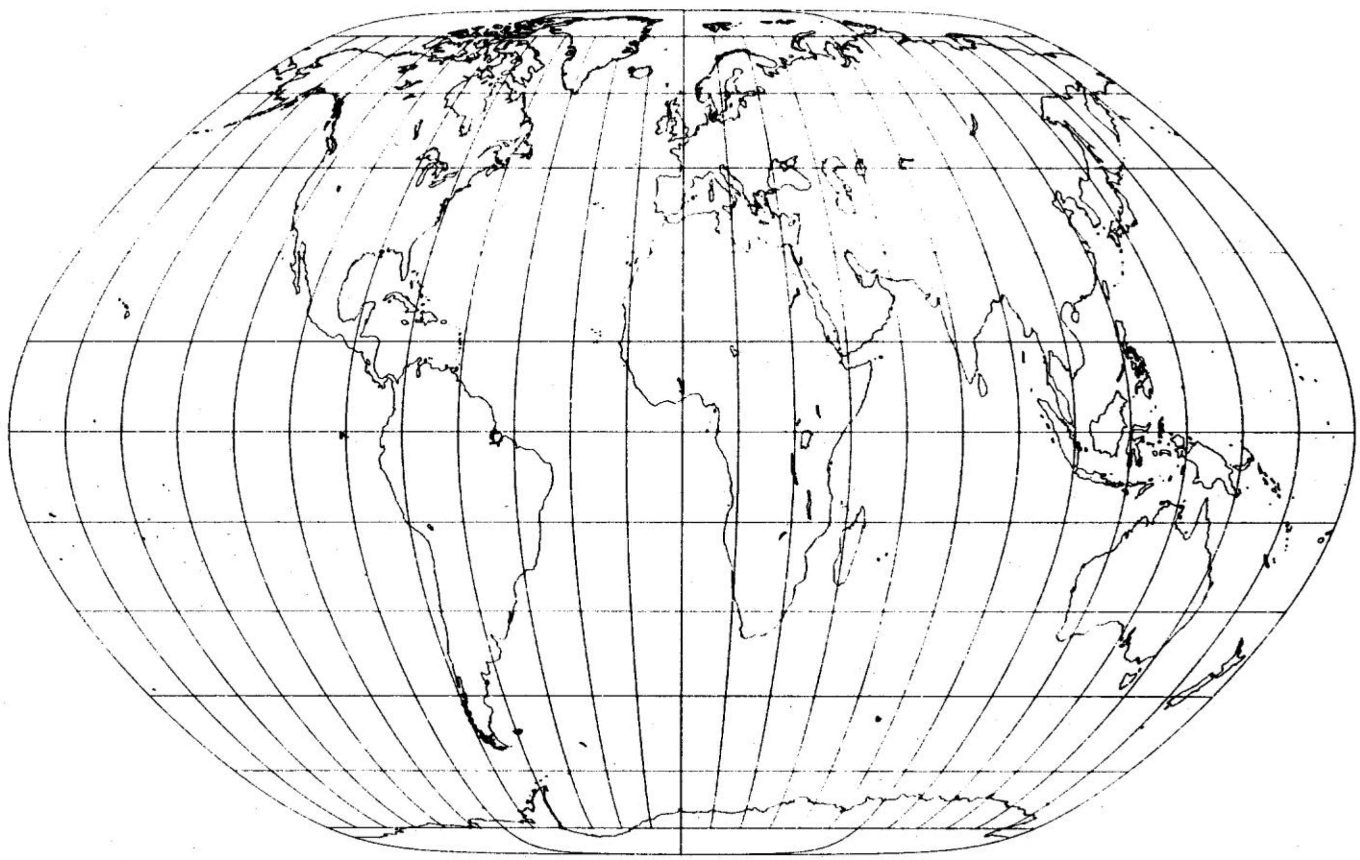
This product can be multiplied additionally:

- by a quadratic function of the geographic latitude φ, which significantly reduces the value of the criterion
- by a quadratic function of the geographic longitude λ, which allows varying scale along the parallels

Six classes of competing functions $x(\varphi,\lambda)$ were investigated.

In line with reduction of the distortions, the outline of a „pointed-polar” world map can approach more and more the outline of a „flat-polar” world map.





Snyder’s „minimum-error” pseudocylindrical equal-area projection
representing the pole as a point

Some world maps of minimum distortions with different kinds of projection equations

Characteristic data given for the presented projections:

- projection equations
- Airy-Kavrayskiy criterion value
- variation of the curvature between the latitudes 85° and 90°

Isolines of distribution of characteristic distortions

- p area scale values

$$p = a \cdot b$$

- 2ω maximum angular deformation values

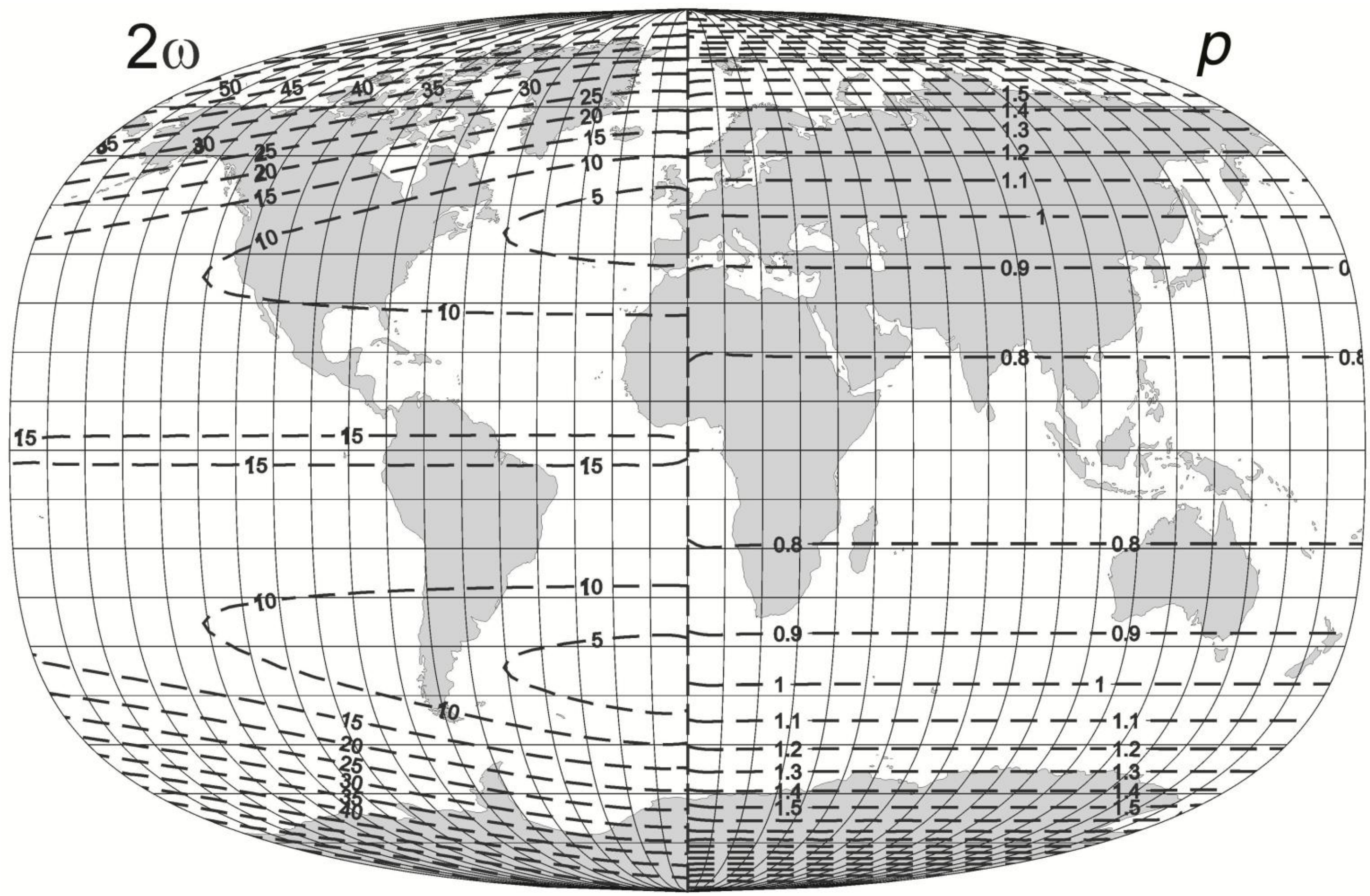
$$2\omega = 2 \cdot \arcsin\left(\frac{a-b}{a+b}\right)$$

$$x = c_1 \cdot \left(1 + c_2 \cdot \varphi^2\right) \cdot \sqrt{1 - \left(\frac{2 \cdot \varphi}{\pi}\right)^2} \cdot \lambda$$

$$y = \varphi$$

$E_K^2 = 0.13339$

$\kappa_{\pm 90^\circ} = 0.134$ $\kappa_{\pm 85^\circ} = 0.221$

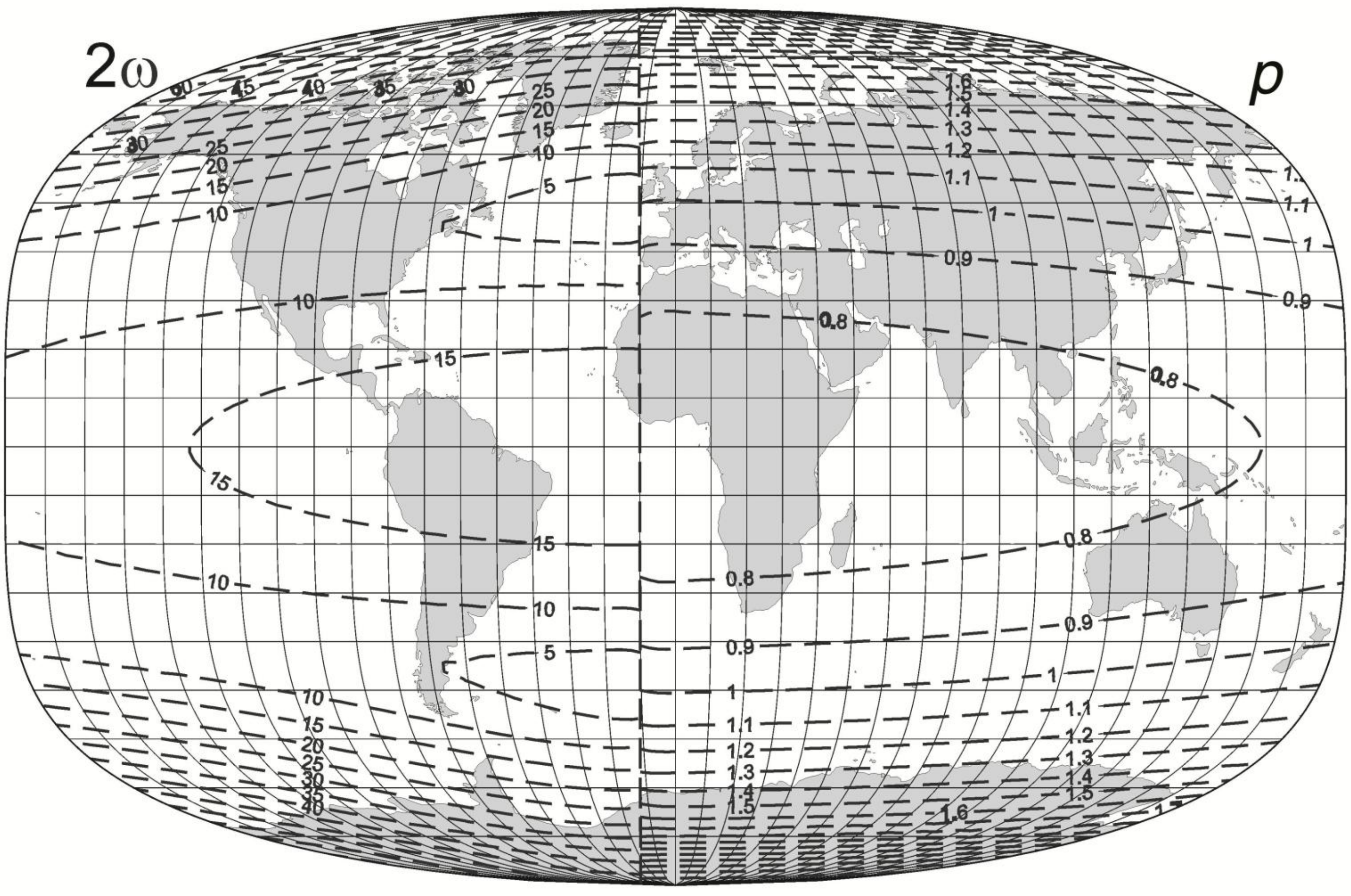


$$x = c_1 \cdot \sqrt{1 - \left(\frac{2 \cdot |\varphi|}{\pi}\right)^{c_2}} \cdot \left(\lambda + c_4 \cdot \lambda^3\right)$$

$$y = \varphi$$

$E_K{}^2 = 0.13510$

$\kappa_{\pm 90°} = 0.129$ $\kappa_{\pm 85°} = 0.211$

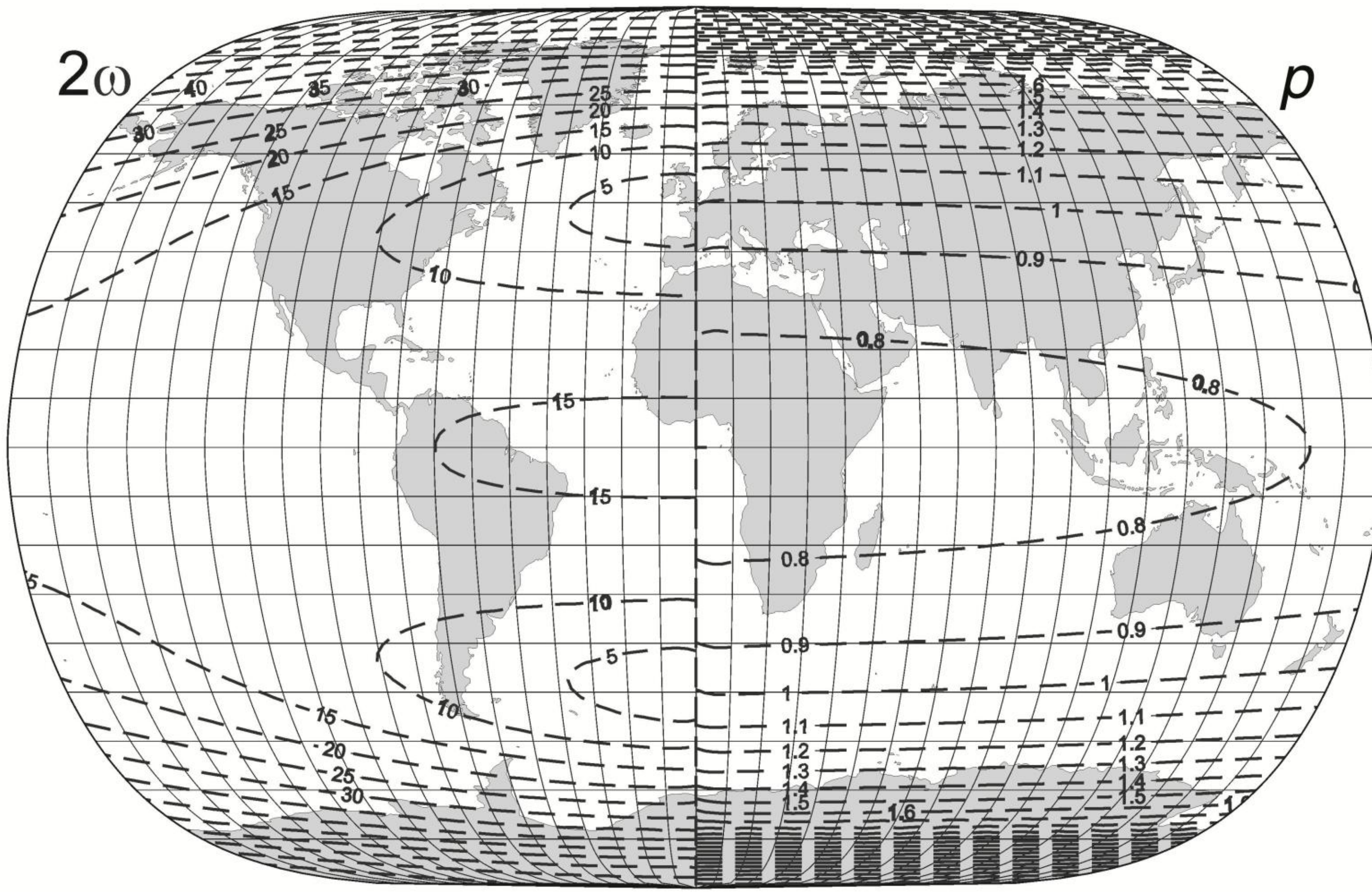


$$x = c_1 \cdot \left[1 - \left(\frac{2 \cdot |\varphi|}{\pi} \right)^{c_2} \right]^{1/c_3} \cdot \left(\lambda + c_4 \cdot \lambda^3 \right)$$

$$y = \varphi$$

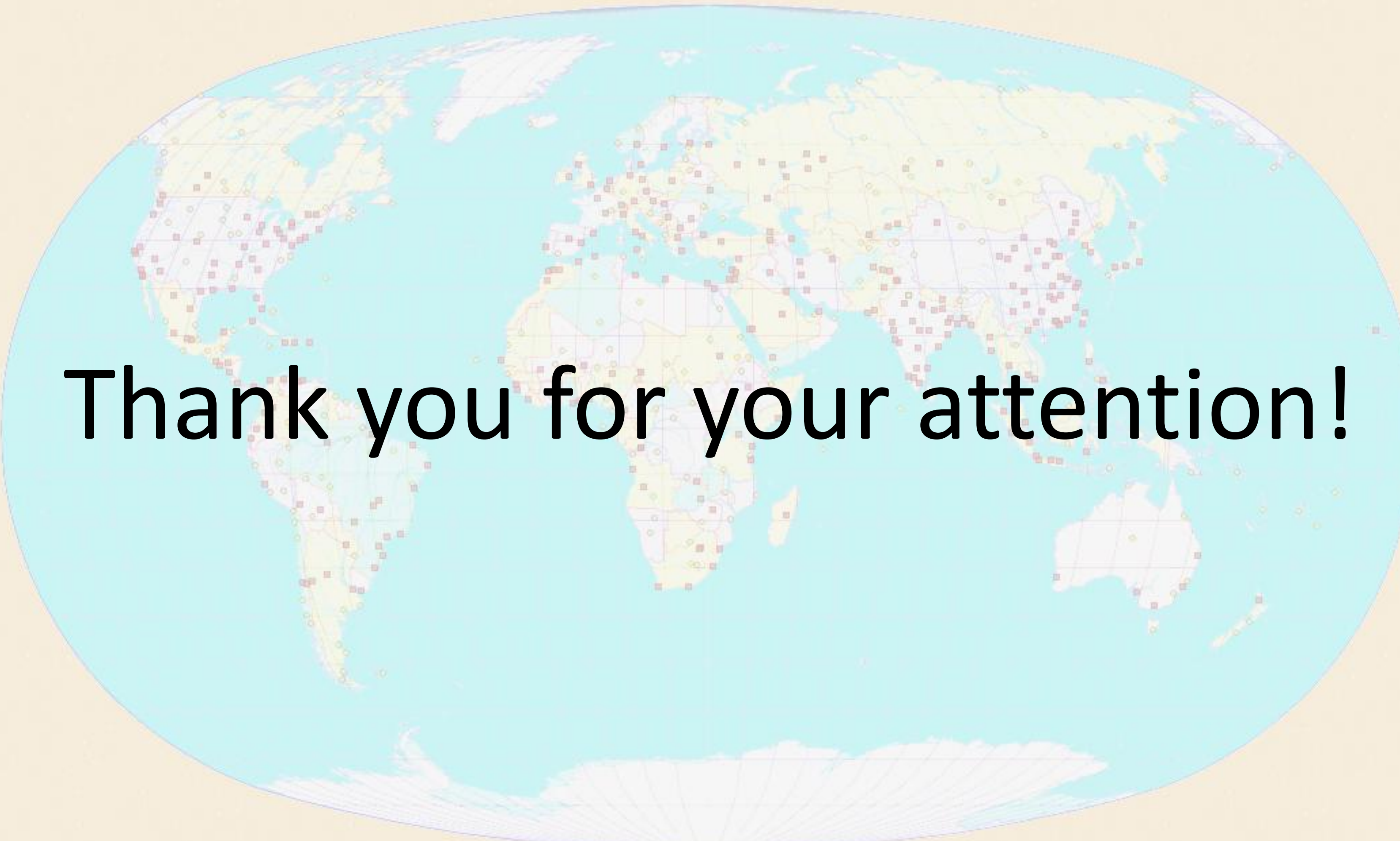
$E_K^2 = 0.12379$

$\kappa_{\pm 90°} = 0.0 \quad \kappa_{\pm 85°} = 0.745$



Conclusions

- the minimum distortion pseudo-cylindrical projection has a *true scale midmeridian*
- its Equator is diminished by about one-fourth
- the linear scale along the parallels *rises* from the midmeridian towards the border meridians
- its outline of the mapped Earth in the pole has zero curvature limit
- other investigated projections with less advantageous mean overall distortion *can keep better the oval shape*



Thank you for your attention!

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> diff(f(yff),yff);
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$$\left(2\,\frac{\ln\left(\frac{1}{2}\sqrt{\frac{xll^2}{\cos(ff)^2}+yff^2+xff^2+2\,\frac{xll\,yff}{\cos(ff)}}+\frac{1}{2}\sqrt{\frac{xll^2}{\cos(ff)^2}+yff^2+xff^2-2\,\frac{xll\,yff}{\cos(ff)}}\right)\left(\frac{1}{4}\,\frac{2\,yff+2\,\frac{xll}{\cos(ff)}}{\sqrt{\frac{xll^2}{\cos(ff)^2}+yff^2+xff^2+2\,\frac{xll\,yff}{\cos(ff)}}}+\frac{1}{4}\,\frac{2\,yff-2\,\frac{xll}{\cos(ff)}}{\sqrt{\frac{xll^2}{\cos(ff)^2}+yff^2+xff^2-2\,\frac{xll\,yff}{\cos(ff)}}}\right)}{\frac{1}{2}\sqrt{\frac{xll^2}{\cos(ff)^2}+yff^2+xff^2+2\,\frac{xll\,yff}{\cos(ff)}}+\frac{1}{2}\sqrt{\frac{xll^2}{\cos(ff)^2}+yff^2+xff^2-2\,\frac{xll\,yff}{\cos(ff)}}}\right.$$
$$\left.+\,2\,\frac{\ln\left(\frac{1}{2}\sqrt{\frac{xll^2}{\cos(ff)^2}+yff^2+xff^2+2\,\frac{xll\,yff}{\cos(ff)}}-\frac{1}{2}\sqrt{\frac{xll^2}{\cos(ff)^2}+yff^2+xff^2-2\,\frac{xll\,yff}{\cos(ff)}}\right)\left(\frac{1}{4}\,\frac{2\,yff+2\,\frac{xll}{\cos(ff)}}{\sqrt{\frac{xll^2}{\cos(ff)^2}+yff^2+xff^2+2\,\frac{xll\,yff}{\cos(ff)}}}-\frac{1}{4}\,\frac{2\,yff-2\,\frac{xll}{\cos(ff)}}{\sqrt{\frac{xll^2}{\cos(ff)^2}+yff^2+xff^2-2\,\frac{xll\,yff}{\cos(ff)}}}\right)}{\frac{1}{2}\sqrt{\frac{xll^2}{\cos(ff)^2}+yff^2+xff^2+2\,\frac{xll\,yff}{\cos(ff)}}-\frac{1}{2}\sqrt{\frac{xll^2}{\cos(ff)^2}+yff^2+xff^2-2\,\frac{xll\,yff}{\cos(ff)}}}\right)\cos(ff)$$