# Mollweide Projection and the ICA Logo 

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#### Abstract

The paper starts with the description of Mollweide's life and work. The formula or equation in mathematics known after him as Mollweide's formula is shown, as well as its proof "without words". Then, the Mollweide map projection is defined and formulas derived in different ways to show several possibilities that lead to the same result. A generalization of Mollweide projection is derived enabling to obtain a pseudocylindrical equal-area projection having the overall shape of an ellipse with any prescribed ratio of its semiaxes. The inverse equations of Mollweide projection has been derived, as well.

The most important part in research of any map projection is distortion distribution. That means that the paper continues with the formulas and images enabling us to get some filling about the liner and angular distortion of the Mollweide projection.

Finally, the ICA logo is used as an example of nice application of the Mollweide projection. A small warning is put on the map painted on the ICA flag. It seams that the map is not produced according to the Mollweide projection and is different from the ICA logo map.


Keywords: Mollweide, Mollweide's formula, Mollweide map projection, ICA logo

## 1. Introduction

Pseudocylindrical map projections have in common straight parallel lines of latitude and curved meridians. Until the 19th century the only pseudocylindrical projection with important properties was the sinusoidal or Sanson-Flamsteed. The sinusoidal has equally spaced parallels of latitude, true scale along parallels, and equivalency or equal-area. As a world map, it has disadvantage of high shear at latitudes near the poles, especially those farthest from the central meridian.

In 1805, Karl Brandan Mollweide (1774-1825) announced an equal-area world map projection that is aesthetically more pleasing than the sinusoidal because the world is placed in an ellipse with axes in a $2: 1$ ratio and all the meridians are equally spaced semiellipses. The Mollweide projection was the only new pseudocylindrical projection of the nineteenth century to receive much more than academic interest (Snyder, 1993).

## 2. Karl Brandan Mollweide

Karl Mollweide was born and brought up in Wolfenbüttel about 15 km south of Brunswick (which today is the city of Braunschweig) in 1774. Unlike most people who become leading mathematicians, Mollweide showed no interest in or talent for the subject while in elementary school. His interest came about very suddenly when he was twelve years old and seems to
have occurred when he discovered some old mathematics books in his home and began reading them. From these books he taught himself calculus and then progressed to the study of algebra. When he was fourteen years old he put his new mathematical skills into practice and calculated the occurrence of an eclipse. This made his mathematical skills more widely known, and his teacher at the gymnasium, Christian Leiste, realised that his pupil was extremely talented (URL1).

With a love of mathematics and a considerable talent for the subject, it was natural that Mollweide would want to study the subject to a higher level. He entered the University of Helmstedt where he was taught by Johann Friedrich Pfaff. Pfaff had much in common with Mollweide for both had studied mathematics largely on their own. Both had, in addition to a deep love of mathematics, an interest in applying it to astronomy. Mollweide entered the University of Helmstedt in 1793 and spent three years studying there. Pfaff was an excellent teacher and at that time was working hard to build up a flourishing mathematics department so Mollweide was happy to be offered a teaching position at the university following his undergraduate studies. However, despite Pfaff's hard work in building up mathematics department, the University of Helmstedt was by this time under threat of closure. This was not the only problem faced by the new lecturer Mollweide for he was suffering severe health problems (probably caused by depression) which forced him to give up his position at Helmstedt after about a year and return to his home.

Back home Mollweide took it easy and spent the next two years essentially taking a prolonged rest. By this time his health improved sufficiently for him to consider accepting an offer of a professorship of mathematics and astronomy at the University of Halle. He decided that his health was now good enough for him to accept the professorship and he took up the position in 1800. He spent eleven years at Halle and it was during this period that he did the two pieces of work for which he is mostly remembered today.

The first of these was his invention of the Mollweide pseudocylindrical projection of the sphere. The second piece of work to which Mollweide's name is attached today is the Mollweide equations which are sometimes called Mollweide's formulas.

In 1811 Mollweide left Halle when he was named Professor of Astronomy at the University of Leipzig. He immediately had an important influence of one of the first students he taught at Leipzig, namely August Möbius. At this time Möbius was intending to make a career as an astronomer but after being taught by Mollweide he became, like his teacher, equally interested in both mathematics and astronomy. As well as being Professor of Astronomy at Leipzig, Mollweide was also director of the university observatory. However, times were difficult because of wars which affected the district. After Napoleon withdrew his armies from Russia in 1812 he began a new offensive against the German states. However his armies failed in their attempt to capture Berlin and retreated to the west. Napoleon's lines of communication were through Leipzig and the allies concentrated their attacks on that point in October 1813. The resulting Battle of Leipzig took place from 16th to 19th October and saw a major defeat for Napoleon. All this concentration on war had made Mollweide's life extremely difficult for he was forced to concentrate on geographic studies to assist the war effort. On top of this, money had to be diverted to the war effort and so Leipzig Observatory received very little funding and could not carry out a proper astronomical research programme.

Mollweide, always more enthusiastic towards mathematics than astronomy, decided in 1814 to move from being Professor of Astronomy to Professor of Mathematics, still at the

University of Leipzig. Certainly the problems in carrying out his duties as Professor of Astronomy had been a big factor in his decision. The chair of astronomy at Leipzig which he vacated was filled by Möbius two years later. The year of 1814 not only marked Mollweide's move from astronomy to mathematics, but it was also the year he married. His wife was the widow of the astronomer Meissner, who had worked at Leipzig Observatory. From 1820 to 1823 Mollweide was Dean of the Leipzig University Faculty of Philosophy.

We referred above to the two contributions for which Mollweide is best remembered today. However he made many other minor contributions published in the Zach's Monatliche Correspondenz (1802-13), in the Zeitsrchrifte für Astronomie (1816-17), in the Gilbert's Annalen der Physik (1804-23), and in the Astronomische Nachrichten (1824-25). Among his other works we mention two published while he was working in Halle: Prüfung der Farbenlehre des Hernn von Goethe (1810), and Darstellung der optischen Irrthümer in Herrn von Goethe's Farbenlehre on a similar topic, which he published in the following year. After moving to Leipzig he published: Commentationes mathematico-philologicae tres (1813); De Quadratis Magicis Commentatio (1816) on magic squares (the first book on the topic not to contain any mysticism); and Adversus grairssimos chronologio myslicae autores (1821). He was also the editor of Euklid's Elements.

Wu says that Mollweide's health problems were basically caused by depression, making him a hypochondriac, and this made him appear standoffish. However, when one got to know him well, one realised what a kind and considerate man he was:
As a teacher, he tried with his full heart to promote the study of science and mathematics. Anyone who expressed interest in these topics received his support. ... he was loved by those who knew him well; deep down he was truly kind and always wanted only the best for science and mathematics. Mollweide was admired as a lecturer because of his ability to present dry topics in an interesting manner by drawing connections to other topics. He was also known for his penmanship; his ability to draw a "perfect" circle freehand amazed his students.

Among his other mathematical accomplishments, Wu mentions:
Mollweide demonstrated various talents as a mathematician. He was feared as a proofreader for his ability to easily detect and harshly criticize the smallest flaw in papers. Although he did not discover any completely new mathematical methods, he was admired for thoroughly investigating and extending known methods. Among his mathematical contributions, Mollweide was the first to use the modern congruence symbol in the 1824 edition of Lorenz's German translation of 'Euklid's Elemente'. He also took over the work on the mathematical dictionary, 'Mathematisches Wörterbuch', from Georg Simon Klügel, but only published one volume in 1823 prior to his death.

## 3. Mollweide's Formulas

In trigonometry, Mollweide's formula, sometimes referred to in older texts as Mollweide's equations, named after Karl Mollweide, is a set of two relationships between sides and angles in a triangle. It can be used to check solutions of triangles.

Let $a, b$, and $c$ be the lengths of the three sides of a triangle. Let $\alpha, \beta$, and $\gamma$ be the measures of the angles opposite those three sides respectively. Mollweide's formulas state that

$$
\frac{a+b}{c}=\frac{\cos \left(\frac{\alpha-\beta}{2}\right)}{\sin \frac{\gamma}{2}} \text { and } \frac{a-b}{c}=\frac{\sin \left(\frac{\alpha-\beta}{2}\right)}{\cos \frac{\gamma}{2}} .
$$

Each of these identities uses all six parts of the triangle - the three angles and the lengths of the three sides.

These trigonometric identities appear in Mollweide's paper Zusätze zur ebenen und sphärischen Trigonometrie (1808). A proof of these identities and an interesting discussion concerning them is given in Wu and a proof without words (see Fig. 1) in DeKleine (1988) and Nelsen (1993).


Fig. 1. Mollweide equations - Proof without Words. According DeKleine (1988).
One of the more puzzling aspects is why these equations should have become known as the Mollweide equations since in the 1808 paper in which they appear Mollweide refers the book by Antonio Cagnoli (1743-1816) Traité de Trigonométrie Rectiligne et Sphérique, Contenant des Méthodes et des Formules Nouvelles, avec des Applications à la Plupart des Problêmes de l'astronomie (1786) which contains the formulas. However, the formulas go back to Isaac Newton, or even earlier, but there is no doubt that Mollweide's discovery was made independently of this earlier work (URL1).

## 4. Mollweide Map Projection Equations

Pseudocylindrical map projections have in common straight parallel lines of latitude and curved meridians. Until the 19th century the only pseudocylindrical projection with important properties was the sinusoidal or Sanson-Flamsteed. The sinusoidal has equally spaced parallels of latitude, true scale along parallels, and equivalency or equal-area. As a world map, it has disadvantage of high distortion at latitudes near the poles, especially those farthest from the central meridian (Fig. 2).


Fig. 2. Sanson or Sanson-Flamsteed or Sinusoidal projection
In 1805 , Mollweide announced an equal-area world map projection that is aesthetically more pleasing than the sinusoidal because the world is placed in an ellipse with axes in a $2: 1$ ratio and all the meridians are equally spaced semiellipses. The Mollweide projection was the only new pseudocylindrical projection of the nineteenth century to receive much more than academic interest (Fig. 3).


Fig. 3. Mollweide projection
Mollweide presented his projection in response to a new globular projection of a hemisphere, described by Georg Gottlieb Schmidt (1768-1837) in 1803 and having the same arrangement of equidistant semiellipses for meridians. But Schmidt's curved parallels do not provide the equal-area property that Mollweide obtained (Snyder, 1993).

O'Connor and Robertson (URL1) stated that Mollweide produced the map projection to correct the distortions in the Mercator projection, first used by Gerardus Mercator in 1569.

While the Mercator projection is well adapted for sea charts, its very great exaggeration of land areas in high latitudes makes it unsuitable for most other purposes. In the Mercator projection the angles of intersection between the parallels and meridians, and the general configuration of the land, are preserved but as a consequence areas and distances are increasingly exaggerated as one moves away from the equator. To correct these defects, Mollweide drew his elliptical projection; but in preserving the correct relation between the areas he was compelled to sacrifice configuration and angular measurement.

The Mollweide projection lay relatively dormant until J. Babinet reintroduced it in 1857 under the name homalographic. The projection has been also called the Babinet, homalographic, homolographic and elliptical projection. It is discussed in many articles, see for example Boggs (1929), Close (1929), Feeman (2000), Philbrick (1953), Reeves (1904) and Snyder (1977) and books or textbooks by Fiala (1957), Graur (1956), Kavrajskij (1960), Kuntz (1990), Maling (1980), Snyder (1987, 1993), Solov'ev (1946) and Wagner (1949).

The well known equations of the Mollweide projections read as follows:

$$
\begin{align*}
& x=\sqrt{2} R \sin \beta  \tag{4.1}\\
& y=\frac{2 \sqrt{2}}{\pi} R \lambda \cos \beta  \tag{4.2}\\
& 2 \beta+\sin 2 \beta=\pi \sin \varphi . \tag{4.3}
\end{align*}
$$

In these formulas $x$ and $y$ are rectangular coordinates in the plane of projection, $\varphi$ and $\lambda$ are geographic coordinates of the points on the sphere and $R$ is the radius of the sphere to be mapped. The angle $\beta$ is an auxiliary angle that is connected with the latitude $\varphi$ by the relation (4.3). For given latitude $\varphi$, the equation (4.3) is a transcendental equation in $\beta$. In the past, it was solving by using tables and interpolation method. In our days, it is usually solved by using some iterative numerical method, like bisection or Newton-Raphson method.

### 4.1. First approach

A half of the sphere with the radius $R$ should be mapped onto the disk with the radius $\rho$ (adopted from Borčić, 1955). If we request that the area of the hemisphere is equal to the area of the disk, than there is the following relation:
$2 R^{2} \pi=\rho^{2} \pi$
from where we have

$$
\begin{equation*}
\rho=\sqrt{2} R . \tag{4.5}
\end{equation*}
$$

Let the circle having the radius $\rho$ be the image of the meridians with the longitudes $\lambda= \pm \frac{\pi}{2}$. From Fig. 4 we see that the rectangular coordinates $x_{0}$ and $y_{0}$ of any point $T_{0}$ belonging to this circle can be written like this:

$$
\begin{equation*}
x_{0}=\rho \sin \beta \tag{4.6}
\end{equation*}
$$

$y_{0}=\rho \cos \beta$


Fig. 4. Derivation of Mollweide projection equations
Due to the request that the projection should be pseudocylindrical, the abscise $x=x_{0}$ for any point with the same latitude regardless of the longitude should be
$x=\sqrt{2} R \sin \beta$.
On the other hand, the ordinate $y$ will depend on the latitude and longitude. According to the equal-area condition, the following relation exists:
$y_{0}: y=\frac{\pi}{2}: \lambda$.

By using (4.9) and (4.5), the relation (4.7) goes into
$y=\frac{2 \sqrt{2}}{\pi} R \lambda \cos \beta$.
In order to finish the derivation, we need to find the relation between the auxiliary angle $\beta$, and the latitude $\varphi$. According to the equal-area condition, the area $S E E_{1} T_{0}$ should be equal to the area of the spherical segment between the equator and the parallel of latitude $\varphi$, which is mapped as the straight-line segment $S T_{0}$ :
$\Delta O S T_{0}+2 O T_{0} E_{1}=R^{2} \pi \sin \varphi$,
that is
$\frac{\rho^{2}}{2} \sin (\pi-2 \beta)+\frac{2 \rho}{2} \beta \rho=R^{2} \pi \sin \varphi$
from where we have

$$
\begin{equation*}
2 \beta+\sin 2 \beta=\pi \sin \varphi . \tag{4.3}
\end{equation*}
$$

### 4.2. Second approach

Given the earth's radius $R$, suppose the equatorial aspect of an equal-area projection with the following properties:

- A world map is bounded by an ellipse twice broader than tall
- Parallels map into parallel straight lines with uniform scale
- The central meridian is a straight standard line; all other ones are semielliptical arcs


Fig. 5. Second approach to derivation of Mollweide projection equations
Suppose an earth-sized map; let us define two regions, $S_{1}$ on the map and $S_{2}$ on the earth, both bounded by the equator and a parallel (URL2). The equal-area property can be used to calculate $x$ for given $\varphi$. Given $x$ and $\lambda, y$ can be calculated immediately from the ellipse equation, since horizontal scale is constant.

Equation of ellipse centred in origin, major axis on $y$-axis:
$\frac{x^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}=1$
$y^{2}=b^{2}\left(1-\frac{x^{2}}{a^{2}}\right)$
For $-a \leq x \leq a$
$y=\frac{b}{a} \sqrt{a^{2}-x^{2}}$.
Area between $y$-axis and parallel mapped into $x=x_{1}$

$$
S_{1}=2 \int_{0}^{x_{1}} y d x=2 \frac{b}{a} \int_{0}^{x_{1}} \sqrt{a^{2}-x^{2}} d x
$$

Let $x=a \sin \beta, 0 \leq x \leq a, 0 \leq \beta \leq \frac{\pi}{2}, d x=a \cos \beta d \beta$,
$\int \sqrt{a^{2}-x^{2}} d x=\int \sqrt{a^{2}\left(1-\sin ^{2} \beta\right)} d \beta=a^{2} \int \cos ^{2} \beta d \beta$.

Since $\cos ^{2} \alpha=\frac{1+\cos 2 \alpha}{2}$
$a^{2} \int \cos ^{2} \beta d \beta=a^{2} \int \frac{1+\cos 2 \beta}{2} d \beta=\frac{a^{2}}{2}\left(\int d \beta+\int \cos 2 \beta d \beta\right)=\frac{a^{2}}{2}\left(\beta+\frac{\sin 2 \beta}{2}\right)+C$
$S_{1}=2 \frac{b}{a}\left(\frac{a^{2}}{2}\left(\beta+\frac{\sin 2 \beta}{2}\right)+C\right)_{0}^{\beta}=\frac{a b}{2}(2 \beta+\sin 2 \beta)=2 R^{2}(2 \beta+\sin 2 \beta)$
for some $0 \leq \beta \leq \frac{\pi}{2}$, corresponding to $x_{1}=a \sin \beta$ and because of $a b \pi=4 R^{2} \pi$.
On a sphere, the area between the equator and parallel $\varphi$ is
$S_{2}=2 \pi R h=2 \pi R^{2} \sin \varphi$
$S_{1}=S_{2} \Rightarrow 2 R^{2}(2 \beta+\sin 2 \beta)=2 \pi R^{2} \sin \varphi$
$2 \beta+\sin 2 \beta=\pi \sin \varphi$
The auxiliary angle $\beta$ must be found by interpolation or successive approximation.
Finally, since horizontal scale is uniform, and $a b \pi=4 R^{2} \pi, b=2 a$ and $a=\sqrt{2} R$ we have
$x=\sqrt{2} R \sin \beta$
Due to the relation
$y: \lambda=\frac{b}{a} \sqrt{a^{2}-x^{2}}: \pi$
$y=\frac{2 \lambda}{\pi} \sqrt{2 R^{2}-x^{2}}=2 \sqrt{2 R^{2}-2 R^{2} \sin ^{2} \beta} \frac{\lambda}{\pi}$
$y=2 \sqrt{2} R \frac{\lambda}{\pi} \cos \beta$

### 4.3. Third approach

From the theory of map projections we know that general equations of pseudocylindrical projections have the form:

$$
\begin{align*}
& x=x(\varphi)  \tag{4.10}\\
& y=y(\varphi, \lambda) \tag{4.11}
\end{align*}
$$

Furthermore, for equal-area pseudocylindrical projection holds
$y=\frac{R^{2} \cos \varphi}{\frac{d x}{d \varphi}} \lambda$.

Let us suppose that a half of the sphere have to be mapped onto a disc with the boundary $x^{2}+y^{2}=\rho^{2}$.

In order to have an equal-area mapping of the half of the sphere with the radius $R$ onto a disc with the radius $\rho$ we should have
$2 R^{2} \pi=\rho^{2} \pi$
from where
$\rho^{2}=2 R^{2}$.
That implies
$x^{2}+y^{2}=2 R^{2}$.

Taking into account (4.12) for $\lambda= \pm \frac{\pi}{2}$
$y= \pm \frac{R^{2} \pi}{2} \frac{\cos \varphi}{\frac{d x}{d \varphi}}$
$x^{2}+\frac{R^{4} \pi^{2}}{4} \frac{\cos ^{2} \varphi}{\left(\frac{d x}{d \varphi}\right)^{2}}=2 R^{2}$.
That is a differential equation that could be solved by the method of separation of variables:
$2 \sqrt{2 R^{2}-x^{2}} d x=R^{2} \pi \cos \varphi d \varphi$
where the sign + has been chosen.
After integration we can get

$$
2 \int \sqrt{2 R^{2}-x^{2}} d x=R^{2} \pi \sin \varphi+C
$$

By the appropriate substitution in the integral on the left side, or just looking to any mathematical manual we can get the following:
$2 \cdot \frac{1}{2}\left(x \sqrt{2 R^{2}-x^{2}}+2 R^{2} \arcsin \frac{x}{R \sqrt{2}}\right)=R^{2} \pi \sin \varphi+C$
For $\varphi=0, x=0$ and $C=0$.

Therefore we have
$x \sqrt{2 R^{2}-x^{2}}+2 R^{2} \arcsin \frac{x}{R \sqrt{2}}=R^{2} \pi \sin \varphi$.
By substitution
$x=\sqrt{2} R \sin \beta$
(4.13) goes to

$$
\begin{equation*}
2 \beta+\sin 2 \beta=\pi \sin \varphi \tag{4.3}
\end{equation*}
$$

while (4.12) can be written as
$y=\frac{2 \lambda}{\pi} \sqrt{2 R^{2}-x^{2}}$
or

$$
\begin{equation*}
y=2 \sqrt{2} R \frac{\lambda}{\pi} \cos \beta . \tag{4.2}
\end{equation*}
$$

## Remark 1

Although the applied condition was that a half of the sphere has to be mapped onto a disc, the final projection equations hold for the whole sphere and give its image situated into an ellipse.

## Remark 2

In references, the Mollweide projection is always defined by equations (4.1)-(4.3), which means by using an auxiliary angle or parameter. My equation (4.13) shows that there is no need to use any auxiliary parameter. There exists the direct relation between the $x$-coordinate and the latitude $\varphi$.

## Remark 3

The method applied in this chapter can be applied in derivation of other pseudocylindrical projections, as are e.g. Sanson projection, Collignon projection or even cylindrical equal-area projection.

## 5. Generalization of Mollweide Projection

Let us consider the shape of the Mollweide projection of the whole sphere. From the equations

$$
\begin{align*}
& x=\sqrt{2} R \sin \beta  \tag{4.1}\\
& y=2 \sqrt{2} R \frac{\lambda}{\pi} \cos \beta \tag{4.2}
\end{align*}
$$

by elimination of $\beta$ it is easy to obtain the equation of a meridian in the projection
$\left(\frac{x}{\sqrt{2} R}\right)^{2}+\left(\frac{\pi y}{2 \sqrt{2} R \lambda}\right)^{2}=1$.
It is obvious that for a given $\lambda$ (5.1) is the equation of an ellipse, i.e. it has a form
$\frac{x^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}=1$,
with
$a=\sqrt{2} R, b=\frac{2 \sqrt{2} R \lambda}{\pi}$.
It follows that the semiaxis $a$ is constant, while $b$ depends on the longitude $\lambda$. If we take $\lambda=\pi$, than $b=2 \sqrt{2} R$, and
$a: b=1: 2$.
and that is the ratio of semiaxes in the Mollweide projection. The question arises: is it possible to find out a pseudocylindrical equal-area projection that will give the whole word in an arbitrary ellipse satisfying any given ratio $a: b$ or $b: a$ ?

The answer is yes, and we are going to proof it. Let us denote $\mu=b: a$.
First of all, the area of an ellipse with the semiaxes $a$ and $b=\mu a$ should be equal to the area of the whole sphere:
$a b \pi=\mu a^{2} \pi=4 R^{2} \pi$.
This is equivalent with
$b=\frac{4 R^{2}}{a}, \mu=\frac{4 R^{2}}{a^{2}}$.
Now, the equation of the ellipse with the centre in the origin and with the semiaxes $a$ and $b$ reads
$\frac{x^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}=1$.
by using (5.3), (5.4) goes into

$$
\frac{x^{2}}{a^{2}}+\frac{y^{2}}{\mu^{2} a^{2}}=1, \text { or }
$$

$y^{2}=\mu^{2}\left(a^{2}-x^{2}\right)$.

Furthermore, the projection should be cylindrical and equal-area, which is generally expressed by
$y=\frac{R^{2} \cos \varphi}{\frac{d x}{d \varphi}} \lambda$.
If we substitute (4.12) into (5.5), taking into account that $\lambda=\pi$, after some minor transformation we can get the following differential equation with separated variables
$R^{2} \pi \cos \varphi d \varphi=\mu \sqrt{a^{2}-x^{2}} d x$.
Integral of the left side of the equation is elementary, while for that on the right side we need a substitution
$x=a \sin \beta$.
This leads to the equation
$\pi \cos \varphi d \varphi=4 \cos ^{2} \beta d \beta$
The application of the trigonometric identity
$\cos ^{2} \beta=\frac{1+\cos 2 \beta}{2}$
give us the following differential equation that is ready for integration:
$\pi \cos \varphi d \varphi=2(1+\cos 2 \beta) d \beta$.
After integration, we obtain
$\pi \sin \varphi=2 \beta+\sin 2 \beta+C$,
where $C$ is integration constant. By using the natural conditions $\varphi=0, x=0$ and $\beta=0$ we obtain $C=0$. In that way, the final form of (5.8) is again the known relation
$2 \beta+\sin 2 \beta=\pi \sin \varphi$.
From (5.6) and (5.7) we have
$\frac{d x}{d \varphi}=\frac{a^{2} \pi \cos \varphi}{4 \sqrt{a^{2}-x^{2}}}=\frac{a \pi \cos \varphi}{4 \cos \beta}$
and taking into account (4.12)
$y=\frac{4 R^{2}}{a \pi} \lambda \cos \beta=\mu a \frac{\lambda}{\pi} \cos \beta=2 R \sqrt{\mu} \frac{\lambda}{\pi} \cos \beta$,
while
$x=a \sin \beta=\frac{2 R}{\sqrt{\mu}} \sin \beta$.
Let us summarize:
$x=\frac{2 R}{\sqrt{\mu}} \sin \beta$.
$y=2 R \sqrt{\mu} \frac{\lambda}{\pi} \cos \beta$.
$2 \beta+\sin 2 \beta=\pi \sin \varphi$.
These are equations defining the generalized Mollweide projection onto an ellipse of any given ratio $\mu=b: a$ of its semiaxes.

## Example 1.

Let us take $\mu=1$, that is $a=b$, and we have a bounding circle. According to (5.3) $a=b=2 R$.


Fig. 6. Generalized Mollweide projection onto a disc

## Example 2.

Let us take $\mu=2$, that is $b=2 a$. According to (5.3) $a=\sqrt{2} R, b=2 \sqrt{2} R$, and we are able to recognize the classic Mollweide projection.

## Example 3.

Let us define the ratio $\mu$, by the condition that the linear scale along the equator equals 1 .
From the theory of map projections it is known that the linear scale along parallels is given by
$n=\frac{\sqrt{G}}{R \cos \varphi}$,
where

$$
G=\left(\frac{\partial x}{\partial \lambda}\right)^{2}+\left(\frac{\partial y}{\partial \lambda}\right)^{2}
$$

In our case $x=x(\varphi)$, which means that

$$
\frac{\partial x}{\partial \lambda}=0 .
$$

The condition
$n=1$ for $\varphi=0$
goes to
$\frac{\partial y}{\partial \lambda}=R \cos \varphi=R$.
Now,

$$
\frac{\partial y}{\partial \lambda}=\frac{2 R \sqrt{\mu}}{\pi} \cos \beta=R
$$

and from there and $\beta=0$ due to $\varphi=0$ we have
$\sqrt{\mu}=\frac{\pi}{2}$, or $\mu=\frac{\pi^{2}}{4}$.

Finally, $a=\frac{4 R}{\pi}, b=R \pi$.
$x=\frac{4}{\pi} R \sin \beta$
$y=R \lambda \cos \beta$
$2 \beta+\sin 2 \beta=\pi \sin \varphi$.
It is easy to see that the linear scale in the direction of meridian is also 1 throughout the equator in this version of Mollweide projection. See also Bromley (1965).


Fig. 7. Generalized Mollweide projection without linear distortions along the equator

## 6. Inverse Equations of Mollweide Projection

The inverse equations of any map projections read as follows:
$\varphi=\varphi(x, y)$
$\lambda=\lambda(x, y)$.
The computation of $\varphi$ and $\lambda$ from given $x$ and $y$ in Mollweide projection is straightforward. In fact, for the given $x$ from (4.1) we can get the auxiliary angle $\beta$

$$
\begin{equation*}
\sin \beta=\frac{x}{\sqrt{2} R} \tag{6.1}
\end{equation*}
$$

Then, from (4.3) we have $\varphi$

$$
\begin{equation*}
\sin \varphi=\frac{1}{\pi}(2 \beta+\sin 2 \beta) \tag{6.2}
\end{equation*}
$$

and from (4.2)

$$
\begin{equation*}
\lambda=\frac{\pi y}{2 \sqrt{2} R \cos \beta} . \tag{6.3}
\end{equation*}
$$

## 7. Distribution of Distortions in Mollweide Projection

For the Mollweide projection given by equations (4.1)-(4.3) it can be derived in the straightforward manner:

$$
\begin{aligned}
& \tan \varepsilon=\frac{2 \tan \beta}{\pi} \lambda \\
& m=\frac{\pi \cos \varphi}{2 \sqrt{2} \cos \beta \cos \varepsilon}
\end{aligned}
$$

$n=\frac{2 \sqrt{2} \cos \beta}{\pi \cos \varphi}$
$2 \tan \frac{\omega}{2}=\sqrt{m^{2}+n^{2}-2}$,
where
$\varepsilon$ is defined by $\varepsilon=\theta-\frac{\pi}{2}$, and $\theta$ is the angle between a meridian and a parallel in the plane of projection
$m$ is a linear scale along meridian
$n$ is a linear scale along parallel
$\omega$ is a maximal angular distortion at a point.
The scale of the area $p=1$ by definition.
The distribution of distortion of Mollweide projection has been investigated and represented in tabular and/or graphical form by several authors (Behrmann, 1909, Solov'ev, 1946, Graur, 1956, Fiala, 1957, Maling 1980).

The linear scale along parallels depends on latitude only. The linear scale along meridians depends both on latitude and longitude. The only standard parallels are $40^{\circ} 44^{\prime} 12^{\prime \prime} \mathrm{N}$ and S . The only two points with no distortion are the intersections of the central meridian and standard parallels.


Fig. 8. The Mollweide projection with Tissot's indicatrix of deformation (URL5)


Fig. 9. Mollweide projection for the whole word, showing isograms for maximum angular deformation $\omega$ at $10^{\circ}, 20^{\circ}, 30^{\circ}, 40^{\circ}$ and $50^{\circ}$. Parts of the world map where $\omega>80^{\circ}$ are shown in black (Maling, 1980; Canters and Crols, 2011).

## 8. Some Applications of Mollweide Projection

For those who would like to research the Mollweide projection in more detail, I would recommend the following web-sites: URL2, URL3 and URL6. Although, do it carefully, do to some incorrect statements occurring on the Internet.

Mollweide's projection has been extremely influential. Besides the developments by Goode (URL7), derived works include the interrupted Sinu-Mollweide projection by A. K. Philbrick's (1953), oblique maps like Bartholomew's Atlantis, and simple rescaling by reciprocal factors which preserve its features - e.g., making the equator a standard parallel free of distortion (Bromley, 1965), or making the whole map circular instead of elliptical (W. Tobler) as indicating in the Chapter 5.


Fig. 10. Mollweide projection of the world (URL5)


Fig. 11. Oblique aspect of the Mollweide projection (Solov'ev, 1946, Kavrajskij, 1960)


Fig. 12. The Atlantis Map (Bartholomew, 1948), Transversal aspect of the Mollweide projection (URL3)


Fig. 13. Inferred contours of the geoid (in metres) for the whole word, based upon Kuala's analysis of variations in gravity potential with both latitude and longitude (Maling 1980)


Fig. 14. Sea-surface freon levels measured by the Global Ocean Data Analysis Project. Projected using the Mollweide projection (URL5).


Fig. 15. The Map Room - A weblog about maps (URL8)


Fig. 16. Full-sky image of Cosmic Microwave Background as seen by the Wilkinson Microwave Anisotropy Probe (URL5).

## Remark

The Mollweide and Hammer projections are occasionally confused, since they are both equalarea and share the elliptical boundary; however, the latter design has curved parallels and is not pseudocylindrical.


Fig. 17. Hammer projection (URL 9)
9. ICA Logo and ICA Flag


Fig. 18. ICA logo (URL4)


Fig. 19. ICA flag - The Mollweide projection is obviously not included

## 10. Conclusion

In preparation.

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